ME 4555 - Lecture 31 - Controller design using Bode

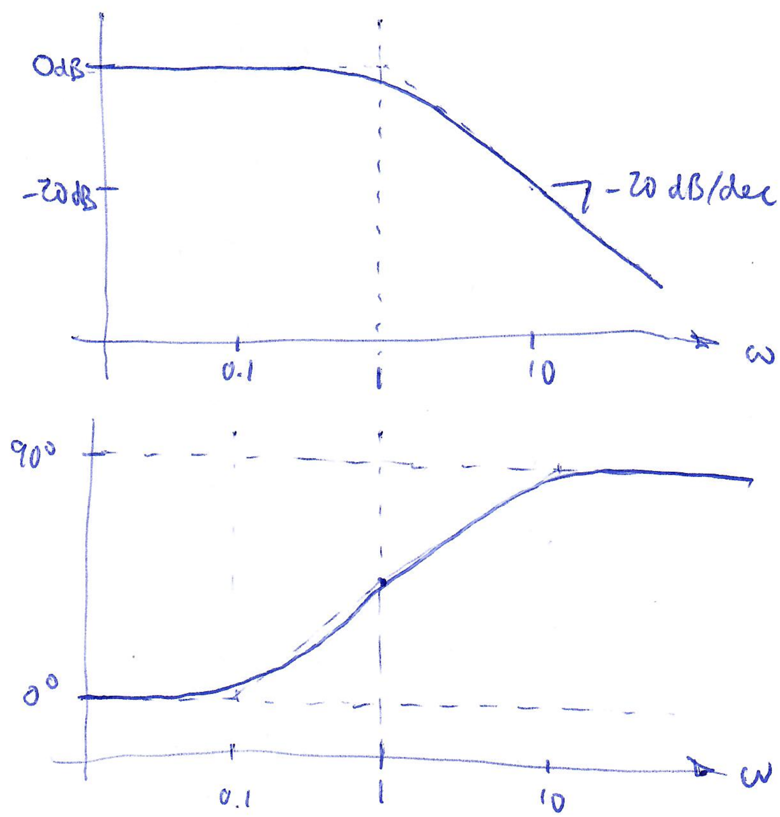
So far, we have used Bode plots to understand the frequency response of stable systems.

But what about unstable systems? We can still plot $M(\omega)$, $\phi(\omega)$ for an unstable system. For example: consider $G(s) = \dfrac{-1}{s-1}$

This is unstable (pole at $s=1$, in the right-half plane), yet we have

$$G(j\omega) = \frac{-1}{j\omega - 1} = \frac{1}{1+\omega^2} + j\cdot\frac{\omega}{1+\omega^2}. \implies \begin{cases} M(\omega) = \dfrac{1}{\sqrt{1+\omega^2}} \\ \phi(\omega) = \arctan\omega \end{cases}$$

so the Bode plot is:



Similar to a 1st order stable pole ($M(\omega)$ is the same as for $\frac{1}{s+1}$) but phase increases instead of decreasing!

However [important]

this Bode plot does not have the same interpretation as when the pole is stable!

Because: $$\underbrace{e^{j\omega t}}_{\text{sinusoidal forcing}} \xrightarrow{\frac{-1}{s-1}} \underbrace{M(\omega)e^{j(\omega t+\phi(\omega))}}_{\text{sinusoidal part}} \underbrace{- \frac{1+j\omega}{\omega^2+1}e^{t}}$$

this "transient" is unstable! So the output of our sinusoidal forcing is actually not a sinusoid in steady-state

Bode plots for unstable poles/zeros.

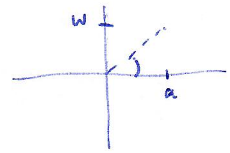
Note that a stable pole $\frac{1}{s+a}$ and an unstable pole $\frac{1}{s-a}$

Have the same magnitude:

$$\left|\frac{1}{jw+a}\right| = \frac{1}{\sqrt{w^2+a^2}}, \quad \left|\frac{1}{jw-a}\right| = \frac{1}{\sqrt{w^2+(-a)^2}} = \frac{1}{\sqrt{w^2+a^2}}.$$

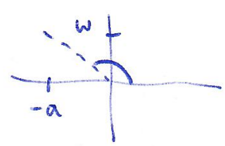
However, the phase is different:

$$\measuredangle\left(\frac{1}{jw+a}\right) = -\measuredangle(jw+a)$$



$$= -\arctan\frac{w}{a}.$$

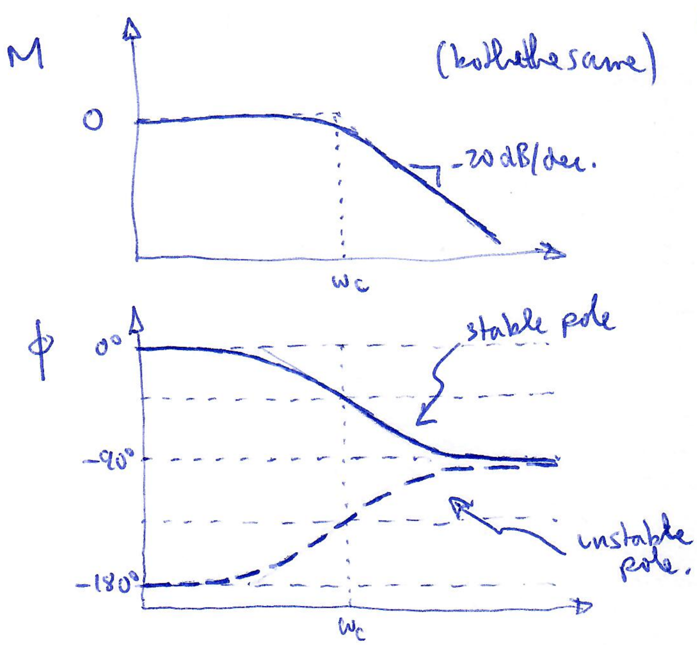
$$\measuredangle\left(\frac{1}{jw-a}\right) = -\measuredangle(jw-a)$$



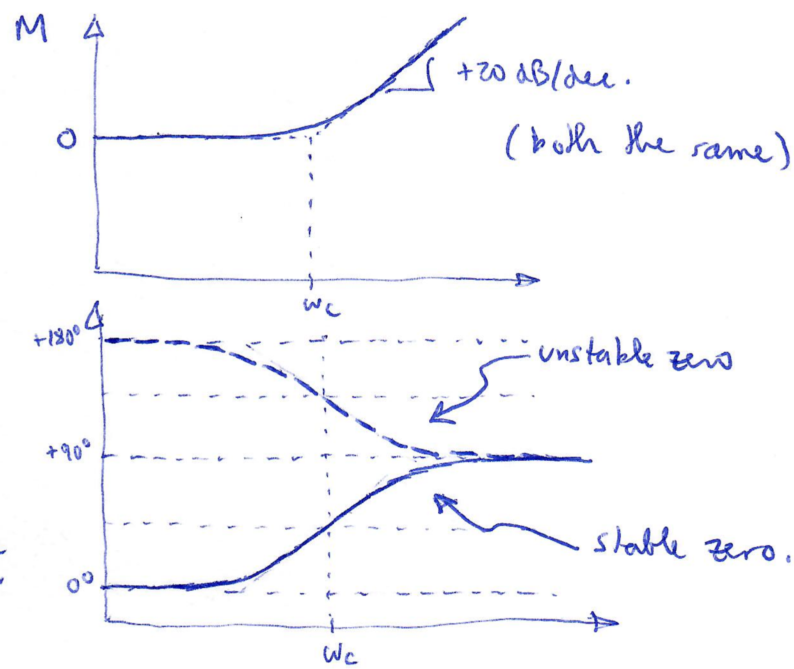
$$= \arctan\frac{w}{a} - \pi$$

Here is what the Bode plot looks like for an unstable/stable pole/zero.

<u>real Pole</u>: $\frac{1}{\frac{s}{w_c} \pm 1}$



<u>real zero</u>: $\left(\frac{s}{w_c} \pm 1\right)$



<u>NOTE</u>: although it's possible to sketch the Bode plot for an unstable system, be careful about how you <u>interpret</u> it! The output to a sinusoidal input will be unbounded (transients do <u>not</u> dissipate)

Unstable zeros add more phase to a system than stable zeros do. This is why unstable zeros are often called "non-minimum phase zeros".

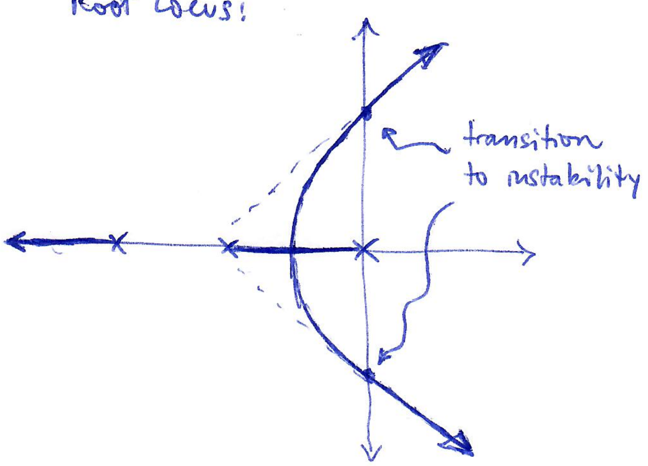
P-controllers using Bode

If our plant is $G(s)$ and we want to design a compensator $C(s) = K$ (P-controller), we want to know about the poles of the closed-loop system.

For now, assume $G(s)$ is stable. Then for small K, the closed loop is stable as well (think about root locus: $\mathcal{C}$ poles start at $\mathcal{O}$ poles. So if those are stable, then $\mathcal{C}$ poles are stable for small enough K). Transition to instability occurs when poles cross the $j\omega$ axis.

Ex: $G(s) = \dfrac{1}{s(s+1)(s+2)}$

Root Locus:



At this critical transition to instability, there are poles that are purely imaginary.

I.e.: denominator of $\dfrac{GK}{1+GK}$ is zero when $s = j\omega$ for some ω.

This means $\boxed{K \cdot G(j\omega) = -1}$

But $G(j\omega)$ is the open loop Bode plot!

So instability occurs when $|K \cdot G(j\omega)| = 1$ and $\angle(K \cdot G(j\omega)) = -180°$

Alternatively, we can write:

Instability occurs when $M(\omega) = \frac{1}{K}$ and $\phi(\omega) = -180°$

So (a) find ω at which $\phi(\omega)$ crosses $-180°$. This is called the "phase crossover frequency". (b) Find $M(\omega)$ at this value of ω.

System is stable for $0 \leq K < \dfrac{1}{M(\omega)}$ $\longrightarrow$ this is called the "gain margin".

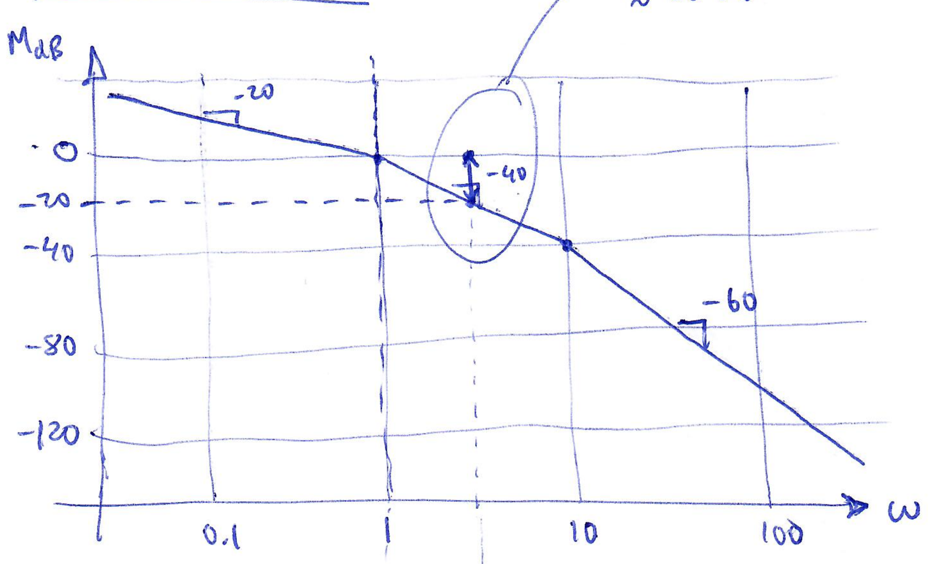
Example: For what values of $C(s) = K$ is the closed-loop with

$$G(s) = \frac{1}{s(s+1)\left(\frac{s}{10}+1\right)}$$

stable?

Draw Bode plot:

gain margin ≈ 20 dB

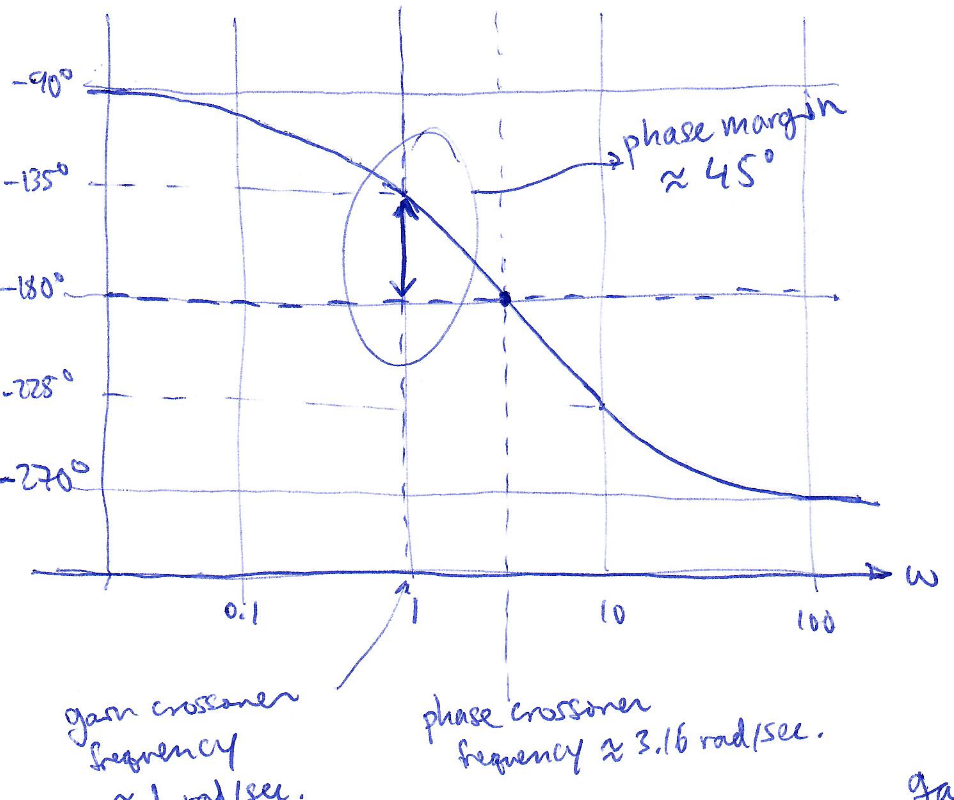


integrator ⇒ initial -20 dB/dec slope.

pole at $s=-1$ ⇒ decrease to -40 dB/dec.

pole at $s=-10$ ⇒ decrease to -60 dB/dec.

$\left\{ \begin{array}{l} -20 \text{ dB/dec per pole;} \\ \text{high-frequency rolloff} = -60 \text{ dB/dec.} \end{array} \right.$



integrator ⇒ initial phase is $-90°$

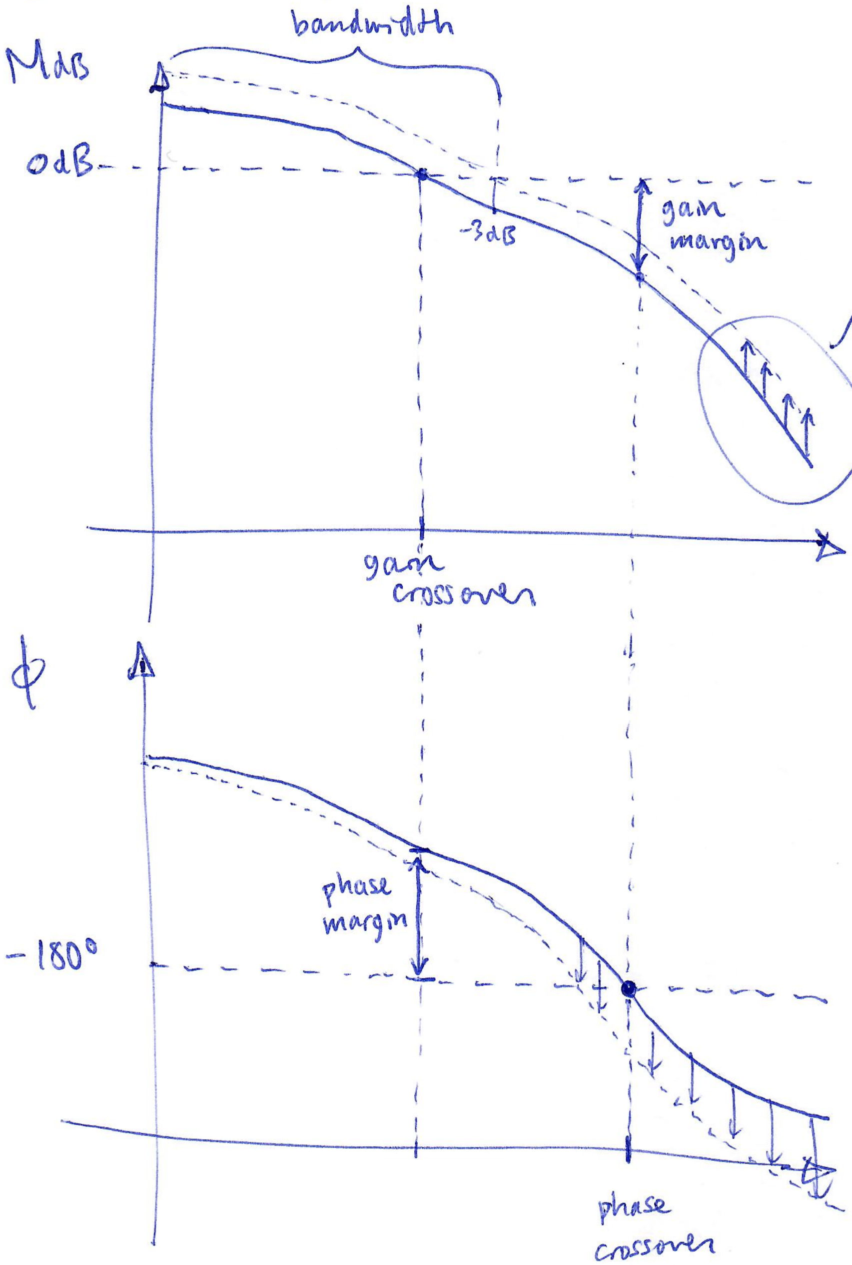
pole at $s=-1$ ⇒ drop $90°$

pole at $s=-10$ ⇒ drop $90°$

$\left\{ \begin{array}{l} -90° \text{ per pole:} \\ \text{high frequency phase shift: } -270°. \end{array} \right.$

Solution at $-180°$ of phase, $M(\omega)$ is approximately -20 dB. So if we use a gain K, this shifts Magnitude plot upwards by $20 \log_{10} K$. We can tolerate a shift of up to 20 dB (i.e. $K = 10$). So system is stable in closed loop for $0 \le K < 10$.

Gain and phase margins tell you about how much of a "safety buffer" you have between your system being stable vs unstable. Consider a typical low-pass filter:



Applying a constant gain K. shifts entire magnitude plot up

$$|K G(j\omega)| = |K| \cdot |G(j\omega)|.$$

$$\Rightarrow \log_{10}|KG(j\omega)| = \underbrace{\log_{10}|K|}_{\text{translation.}} + \log_{10}|G(j\omega)|$$

The gain margin (GM) is the largest gain we can use and still remain stable.

Note: $\mathcal{L}\{f(t-\tau)\} = e^{-s\tau}F(s)$

delay of τ seconds.

So if there is a delay of τ seconds in the loop, we have $G(j\omega)$ becomes $e^{-j\tau\omega}G(j\omega)$

delay will cause a phase shift without affecting the magnitude

phase due to delay is

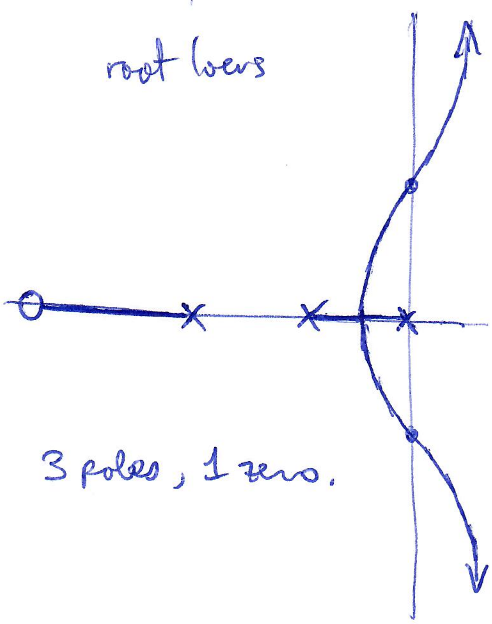
$$\measuredangle\{e^{-j\tau\omega}G(j\omega)\} = \phi(\omega) - \tau\omega.$$

☆ having extra gain + phase margin allows you to be a bit wrong (margin of error) in your plant model and not risk going unstable. It makes your controller <u>more robust</u>.

☆ there is a trade-off. Making K larger gives you more bandwidth (good thing) at the expense of stability ...

Example

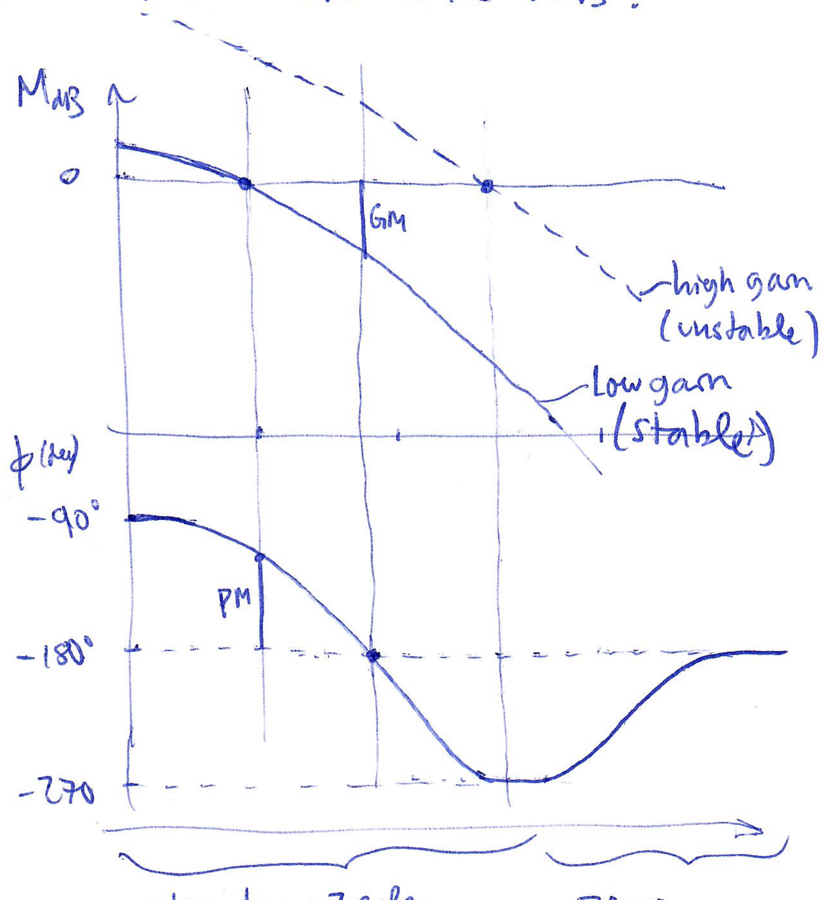
root locus



3 poles, 1 zero.

even though there are 3 poles + 1 zero (vertical asymptotes) this might lead us to think that there is only $(3-1)\cdot(-90) = -180°$ deg of phase and so we can't have instability. Not so! the three poles occur "first".

So the Bode plot looks like this:



adding more gain

- increases the bandwidth
- causes the GM + PM to shrink
- causes the gain crossover and phase crossover frequencies to get closer together
- if K is increased too much, we have gain crossover > phase crossover, which means instability.